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On the phase shift of line solitary waves for the KP-II equation

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1 Introduction

The KP-II equation

$$(1.1) \quad \partial_x(\partial_t u + \partial_x^3 u + 3\partial_x(u^2)) + 3\partial_y^2 u = 0 \quad \text{for } t > 0 \text{ and } (x, y) \in \mathbb{R}^2,$$

is a generalization to two spatial dimensions of the KdV equation

$$(1.2) \quad \partial_t u + \partial_x^3 u + 3\partial_x(u^2) = 0,$$

and has been derived as a model in the study of the transverse stability of solitary wave solutions to the KdV equation with respect to two dimensional perturbation when the surface tension is weak or absent (see [13]).

The global well-posedness of (1.1) in $H^s(\mathbb{R}^2)$ ($s \geq 0$) on the background of line solitons has been studied by Molinet, Saut and Tzvetkov [22] whose proof is based on the work of Bourgain [4]. For the other contributions on the Cauchy problem of the KP-II equation, see e.g. [9, 10, 11, 12, 28, 29, 30, 31] and the references therein.

Let

$$\varphi_c(x) \equiv c \cosh^{-2} \left(\sqrt{\frac{c}{2}} x \right), \quad c > 0.$$

Then φ_c is a solution of

$$(1.3) \quad \varphi_c'' - 2c\varphi_c + 3\varphi_c^2 = 0,$$

and $\varphi_c(x - 2ct)$ is a solitary wave solution of the KdV equation (1.2) and a line soliton solution of (1.1) as well. In this article, we report my recent result on the phase shift of modulating line solitons.

Let us briefly explain known results on stability of 1-solitons for the KdV equation first. Stability of the 1-soliton $\varphi_c(x - 2ct)$ of (1.2) was proved by [2, 3, 32] using the fact that φ_c is a minimizer of the Hamiltonian on the manifold $\{u \in H^1(\mathbb{R}) \mid \|u\|_{L^2(\mathbb{R})} = \|\varphi_c\|_{L^2(\mathbb{R})}\}$.

Solitary waves of the KdV equation travel at speeds faster than the maximum group velocity of linear waves and the larger solitary wave moves faster to the right. Using this property, Pego and Weinstein [24] prove asymptotic stability of solitary wave solutions of (1.2) in an exponentially weighted space. Later, Martel and Merle established the Liouville theorem for the generalized KdV equations by using a virial type identity and prove the asymptotic stability of solitary waves in $H_{loc}^1(\mathbb{R})$ (see e.g. [16]).

For the KP-II equation, its Hamiltonian is infinitely indefinite and the variational characterization of line soliton is not useful. So it is natural to study stability of line solitons using strong linear stability of line solitons as in [24]. Spectral transverse stability of line solitons of (1.1) has

been studied by [1, 5]. Alexander *et al.* [1] proved that the spectrum of the linearized operator in $L^2(\mathbb{R}^2)$ consists of the entire imaginary axis. On the other hand, in an exponentially weighted space where the size of perturbations are biased in the direction of motion, the spectrum of the linearized operator consists of a curve of resonant continuous eigenvalues which goes through 0 and the set of continuous spectrum which locates in the stable half plane and is away from the imaginary axis (see [5, 17]). The former one appears because line solitons are not localized in the transversal direction and 0, which is related to the symmetry of line solitons, cannot be an isolated eigenvalue of the linearized operator. Such a situation is common with planer traveling wave solutions for the heat equation. See e.g. [14, 15, 33].

Transverse stability of line solitons for the KP-II equation has been proved for localized perturbations as well as for perturbations which have 0-mean along all the lines parallel to the x -axis ([17, 18]).

Theorem 1.1. ([18, Theorem 1.1]) *Let $c_0 > 0$ and $u(t, x, y)$ be a solution of (1.1) satisfying $u(0, x, y) = \varphi_{c_0}(x) + v_0(x, y)$. There exist positive constants ε_0 and C satisfying the following: if $v_0 \in \partial_x L^2(\mathbb{R}^2)$ and $\|v_0\|_{L^2(\mathbb{R}^2)} + \| |D_x|^{1/2} v_0 \|_{L^2(\mathbb{R}^2)} + \| |D_x|^{-1/2} |D_y|^{1/2} v_0 \|_{L^2(\mathbb{R}^2)} < \varepsilon_0$ then there exist C^1 -functions $c(t, y)$ and $x(t, y)$ such that for every $t \geq 0$ and $k \geq 0$,*

$$(1.4) \quad \|u(t, x, y) - \varphi_{c(t, y)}(x - x(t, y))\|_{L^2(\mathbb{R}^2)} \leq C \|v_0\|_{L^2},$$

$$(1.5) \quad \|c(t, \cdot) - c_0\|_{H^k(\mathbb{R})} + \|\partial_y x(t, \cdot)\|_{H^k(\mathbb{R})} + \|x_t(t, \cdot) - 2c(t, \cdot)\|_{H^k(\mathbb{R})} \leq C \|v_0\|_{L^2},$$

$$(1.6) \quad \lim_{t \rightarrow \infty} \left(\|\partial_y c(t, \cdot)\|_{H^k(\mathbb{R})} + \|\partial_y^2 x(t, \cdot)\|_{H^k(\mathbb{R})} \right) = 0,$$

and for any $R > 0$,

$$(1.7) \quad \lim_{t \rightarrow \infty} \|u(t, x + x(t, y), y) - \varphi_{c(t, y)}(x)\|_{L^2((x > -R) \times \mathbb{R}_y)} = 0.$$

Let $\langle x \rangle = \sqrt{1 + x^2}$ for $x \in \mathbb{R}$.

Theorem 1.2. ([18, Theorem 1.2]) *Let $c_0 > 0$ and $s > 1$. Suppose that u is a solutions of (1.1) satisfying $u(0, x, y) = \varphi_{c_0}(x) + v_0(x, y)$. Then there exist positive constants ε_0 and C such that if $\|\langle x \rangle^s v_0\|_{H^1(\mathbb{R}^2)} < \varepsilon_0$, there exist $c(t, y)$ and $x(t, y)$ satisfying (1.6), (1.7) and*

$$(1.8) \quad \|u(t, x, y) - \varphi_{c(t, y)}(x - x(t, y))\|_{L^2(\mathbb{R}^2)} \leq C \|\langle x \rangle^s v_0\|_{H^1(\mathbb{R}^2)},$$

$$(1.9) \quad \|c(t, \cdot) - c_0\|_{H^k(\mathbb{R})} + \|\partial_y x(t, \cdot)\|_{H^k(\mathbb{R})} + \|x_t(t, \cdot) - 2c(t, \cdot)\|_{H^k(\mathbb{R})} \leq C \|\langle x \rangle^s v_0\|_{H^1(\mathbb{R}^2)}$$

for every $t \geq 0$ and $k \geq 0$.

Remark 1.1. The parameters $c(t_0, y_0)$ and $x(t_0, y_0)$ represent the local amplitude and the local phase shift of the modulating line soliton $\varphi_{c(t, y)}(x - x(t, y))$ at time t_0 along the line $y = y_0$ and that $x_y(t, y)$ represents the local orientation of the crest of the line soliton.

Remark 1.2. In view of Theorem 1.1,

$$\lim_{t \rightarrow \infty} \sup_{y \in \mathbb{R}} (|c(t, y) - c_0| + |x_y(t, y)|) = 0,$$

and as $t \rightarrow \infty$, the modulating line soliton $\varphi_{c(t, y)}(x - x(t, y))$ converges to a y -independent modulating line soliton $\varphi_{c_0}(x - x(t, 0))$ in $L^2(\mathbb{R}_x \times (|y| \leq R))$ for any $R > 0$.

For the KdV equation as well as for the KP-II equation posed on $L^2(\mathbb{R}_x \times \mathbb{T}_y)$, the dynamics of a modulating soliton $\varphi_{c(t)}(x - x(t))$ is described by a system of ODEs

$$\dot{c} \simeq 0, \quad \dot{x} \simeq 2c.$$

See [24] for the KdV equation and [20] for the KP-II equation with the y -periodic boundary condition. However, to analyze transverse stability of line solitons for localized perturbation in \mathbb{R}^2 , we need to study a system of PDEs for $c(t, y)$ and $x(t, y)$ in [17, 18] as is the case with the planar traveling waves for the heat equations (e.g. [14, 15, 33]) and planar kinks for the ϕ^4 -model ([6]).

In [17, Theorems 1.4 and 1.5], we find that the phase shift $x(t, y)$ in (1.4) and (1.7) is not uniform in y because of the diffraction of modulating line solitons around $y = \pm 2\sqrt{2c_0}t + O(\sqrt{t})$ and that the set of exact 1-line solitons

$$\mathcal{K} = \{\varphi_c(x + ky - (2c + 3k^2)t + \gamma) \mid c > 0, k, \gamma \in \mathbb{R}\}$$

is not stable in $L^2(\mathbb{R}^2)$.

Theorem 1.3. *Let $c_0 > 0$. Then for any $\varepsilon > 0$, there exists a solution of (1.1) such that $\|\langle x \rangle(\langle x \rangle + \langle y \rangle) \{u(0, x, y) - \varphi_{c_0}(x)\}\|_{H^1(\mathbb{R}^2)} < \varepsilon$ and $\liminf_{t \rightarrow \infty} t^{-1/2} \inf_{v \in \mathcal{A}} \|u(t, \cdot) - v\|_{L^2(\mathbb{R}^2)} > 0$.*

Theorem 1.4. *Let $c_0 = 2$ and $u(t)$ be as in Theorem 1.2. There exist positive constants ε_0 and C such that if $\varepsilon := \|\langle x \rangle(\langle x \rangle + \langle y \rangle)v_0\|_{H^1(\mathbb{R}^2)} < \varepsilon_0$, then there exist C^1 -functions $c(t, y)$ and $x(t, y)$ satisfying (1.4)–(1.7) and*

$$(1.10) \quad \left\| \begin{pmatrix} c(t, \cdot) - 2 \\ x_y(t, \cdot) \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_B^+(t, y + 4t) \\ u_B^-(t, y - 4t) \end{pmatrix} \right\|_{L^2(\mathbb{R})} = o(\varepsilon t^{-1/4})$$

as $t \rightarrow \infty$, where u_B^\pm are self similar solutions of the Burgers equation

$$\partial_t u = 2\partial_y^2 u \pm 4\partial_y(u^2)$$

such that

$$u_B^\pm(t, y) = \frac{\pm m_\pm H_{2t}(y)}{2(1 + m_\pm \int_0^y H_{2t}(y_1) dy_1)}, \quad H_t(y) = (4\pi t)^{-1/2} e^{-y^2/4t},$$

and that m_\pm are constants satisfying

$$\int_{\mathbb{R}} u_B^\pm(t, y) dy = \frac{1}{4} \int_{\mathbb{R}} (c(0, y) - 2) dy + O(\varepsilon^2).$$

Remark 1.3. Since (1.1) is invariant under the scaling $u \mapsto \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y)$, we may assume that $c_0 = 2$ without loss of generality.

Remark 1.4. The linearized operator around the line soliton solution has resonant continuous eigenvalues near $\lambda = 0$ whose corresponding eigenmodes grow exponentially as $x \rightarrow -\infty$. See (??)–(??). The diffraction of the line soliton around $y = \pm 4t$ can be thought as a mechanism to emit energy from those resonant continuous eigenmodes.

Theorems 1.3 and 1.4 are improvement of [17, Theorems 1.4 and 1.5].

If we disregard damping effect and nonlinearity of waves propagating along the crest of line solitons, then time evolution of the phase shift is expected to be described by the 1-dimensional wave equation

$$x_{tt} = 8c_0 x_{yy}.$$

So it seems natural to expect that $\sup_{t,y \in \mathbb{R}} |x(t,y) - 2c_0 t|$ remains small for localized perturbations even if the $L^2(\mathbb{R}_y)$ norm of $x(t,y) - 2c_0 t$ grows as $t \rightarrow \infty$. We have the following result for the phase shift of modulating 1-line solitons.

Theorem 1.5. *Let $u(t,x,y)$ and $x(t,y)$ be as in Theorem 1.2. There exist positive constants ε_0 and C such that if $\varepsilon := \| \langle x \rangle (\langle x \rangle + \langle y \rangle) v_0 \|_{H^1(\mathbb{R}^2)} < \varepsilon_0$, then $\sup_{t \geq 0, y \in \mathbb{R}} |x(t,y) - 2c_0 t| \leq C\varepsilon$.*

Moreover, there exists an $h \in \mathbb{R}$ such that for any $\delta > 0$,

$$(1.11) \quad \begin{cases} \lim_{t \rightarrow \infty} \|x(t, \cdot) - 2c_0 t - h\|_{L^\infty(|y| \leq (\sqrt{8c_0} - \delta)t)} = 0, \\ \lim_{t \rightarrow \infty} \|x(t, \cdot) - 2c_0 t\|_{L^\infty(|y| \geq (\sqrt{8c_0} + \delta)t)} = 0. \end{cases}$$

In the case where the surface tension is weak, we find in [19] that time evolution of resonant continuous eigenmondes for the linearized Benney-Luke equation around line solitary waves is similar to (1.11). We except that (1.11) is true for modulating line solitary waves of the 2D Benney-Luke equation.

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